Minimal Morse functions via the heat equation in locally homogeneous riemannian manifolds

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(M, g)

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g: a riemannian geometry for M

Laplace-Beltrami

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$\triangle_g$: Laplace-Beltrami operator on (M, g)

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Laplace-Beltrami

$\triangle_g$: Laplace-Beltrami operator on (M, g)

$$\triangle_g = div(grad(\cdot))$$

(Chavel, 1984)

Heat equation on (M, g)

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$$\begin{cases} \dfrac{\partial f}{\partial t} & = & \triangle_g (f) \\ f(\cdot, 0) & = & h \in L^2(M) \end{cases}$$

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$$f : M \times (0, \infty) \to \mathbb{R}$$

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1. for each $t > 0$, $f(\cdot, t)$ is C^2, and for each $x \in M$, $f(x, \cdot)$ is C^1

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such that (Jorgenson and Lang, 2003):

1. for each $t > 0$, $f(\cdot, t)$ is C^2, and for each $x \in M$, $f(x, \cdot)$ is C^1
2. $\dfrac{\partial f}{\partial t} = \triangle_g(f)$ and

$$\lim_{t \to 0^+} \int_M f(x,t)\psi(x) dvol_g = \int_M h(x)\psi(x) dvol_g$$

$$\forall \psi \in C^\infty(M)$$

Solution method

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- $\triangle_g$ has real eigenvalues

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

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and

- $L^2(M)$ has an orthonormal basis

$$\{\varphi_{ij} : i \geq 0, 1 \leq j \leq m_i < \infty\}$$

whit $\triangle_g(\varphi_{ij}) = \lambda_i \varphi_{ij}$

Solution method

- Thus, if $E_{\lambda_i} = span\{\varphi_{i1}, \cdots, \varphi_{im_i}\}$ we have

$$L^2(M) = E_{\lambda_0} \oplus E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots$$

i.e. each $h \in L^2(M)$ can be written uniquely as

$$h = h_0 + h_1 + h_2 + \cdots$$

with $h_i \in E_{\lambda_i}$. (Fourier decomposition of h)

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- Connectedness of M implies $h_0 = c$ is a constant. (Lehoucq et al., 2003)

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is:

$$f(x,t) = c + e^{-\lambda_1 t} h_1(x) + e^{-\lambda_2 t} h_2(x) + \cdots$$

Where $h(x) = c + h_1(x) + h_2(x) + \cdots$ is the Fourier decomposition of h.

Solution to problem

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(M, g) is *locally homogeneous* if for any $p, q \in M$ there are open neighborhoods U, V of p and q, respectively, such that there exists an isometry $\varphi : U \to V$ sending p to q.

Morse functions on M

Morse functions on M

$f : M \to \mathbb{R}$ smooth; $p \in M$ is a *nondegenerate critical point of* f if there are local coordinates $x_1, \cdots, x_n$ such that $x_i(p) = 0$ for all i, and

$$f = f(p) - \sum_{i=1}^{k(p)} x_i^2 + \sum_{i=k(p)+1}^{n} x_i^2$$

where $0 \leq k(p) \leq n$ and

$$\sum_{i=1}^{0} x_i^2 = \sum_{i=n+1}^{n} x_i^2 = 0$$

$k(p)$ is called the index of the critical point p
(Milnor, 2016)

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Definition of Morse function

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We will say that a Morse function on M is minimal if there is no Morse function on M having fewer critical points.

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To what extent is it true that if (M, g) is a locally homogeneous riemannian minifold, there exists an open dense subset U of $L^2(M)$, having the property that for each $h \in U$, there exists $t_h > 0$ such that if $f_t, t \geq 0$, is the solution to

$$\begin{cases} \dfrac{\partial f}{\partial t} & = \triangle_g(f) \\ f(x,0) & = h \end{cases}$$

$f_t : M \to \mathbb{R}$ is a minimal Morse function for each $t \geq t_h$?

Simplest case:

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Suppose that the functions in the first nontrivial eigenspace E_{λ_1} of (M, g) are generically minimal Morse functions of M, i.e. that the set

$$\{(a_1, \cdots, a_{m_i}) : a_1\varphi_{11} + \cdots + a_{m_1}\varphi_{1m_1} \text{ is minimal Morse}\}$$

is open and dense in $\mathbb{R}^{m_1}$.

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and then

$$\left\| e^{\lambda_1 t} (f_t - c) - h_1 \right\|_{C^\infty} = \left\| e^{(\lambda_1 - \lambda_2) t} h_2 + \cdots \right\|_{C^\infty}$$

The right hand side goes to zero as $t \to \infty$.

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Simplest case:

So,

$$\left\| e^{\lambda_1 t} \left(f_t - c \right) - h_1 \right\| \to 0 \text{ so } t \to \infty$$

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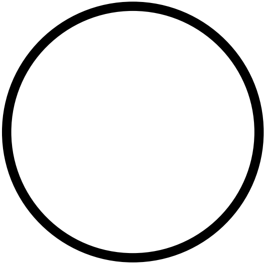
Now, since h_1 is minimal Morse, a direct consequence of Mather's stability theorem implies that $e^{\lambda_1 t} \left(f_t - c \right)$ is minimal Morse for large enought t, which turn implies that $f_t - c$ is a minimal Morse function for large enough t.

EXAMPLES

Circle S^1

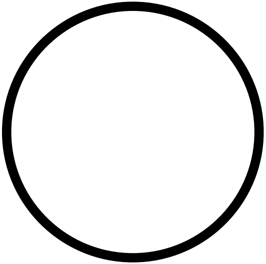
Circle S^1



Circle S^1

$$\lambda_1 = 1, \quad \lambda_2 = 4, \quad \lambda_3 = 9, \quad \lambda_4 = 16, \cdots$$

$$\begin{aligned} E_{\lambda_1} =& span\left\{\sin(x), \cos(x)\right\} \\ E_{\lambda_2} =& span\left\{\sin(2x), \cos(2x)\right\} \\ E_{\lambda_3} =& span\left\{\sin(3x), \cos(3x)\right\} \\ E_{\lambda_4} =& span\left\{\sin(4x), \cos(4x)\right\} \\ &\vdots \end{aligned}$$



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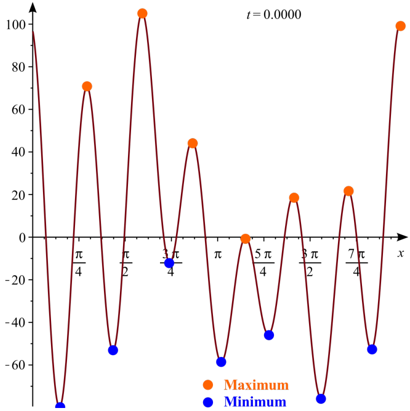
$$f(x,t) = c + [a_1 \sin(x) + b_1 \cos(x)]\, e^{-t} + [a_2 \sin(2x) + b_2 \cos(2x)]\, e^{-4t} + [a_3 \sin(3x) + b_3 \cos(3x)]\, e^{-9t} + [a_4 \sin(4x) + b_4 \cos(4x)]\, e^{-16t} + \cdots$$



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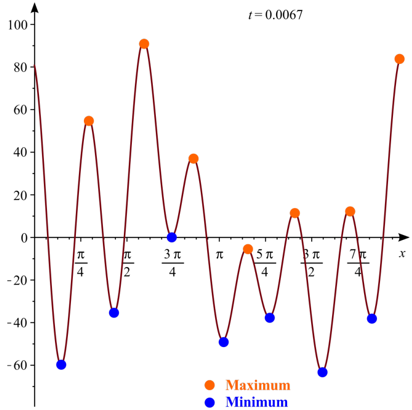

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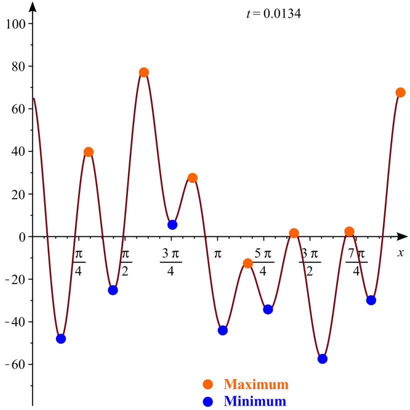

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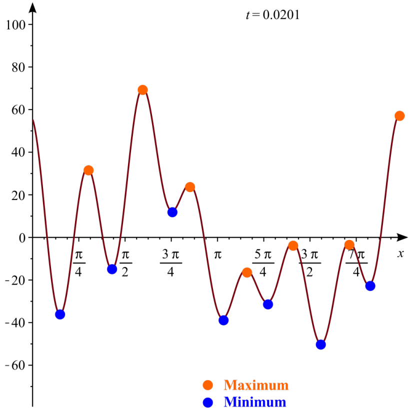

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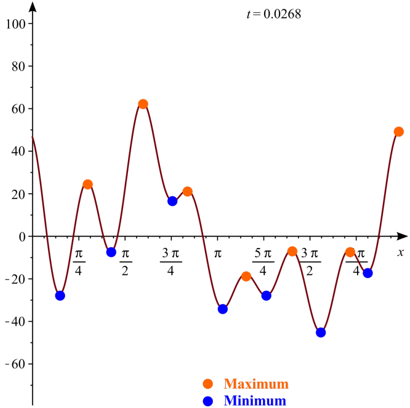

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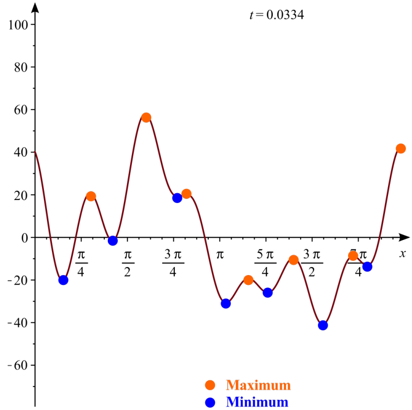

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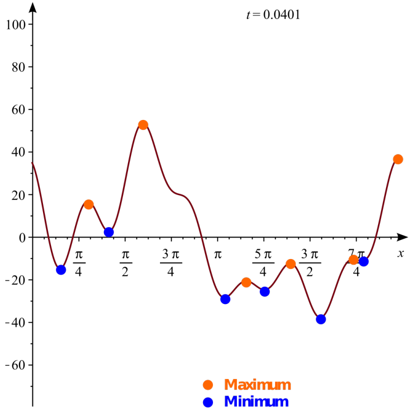

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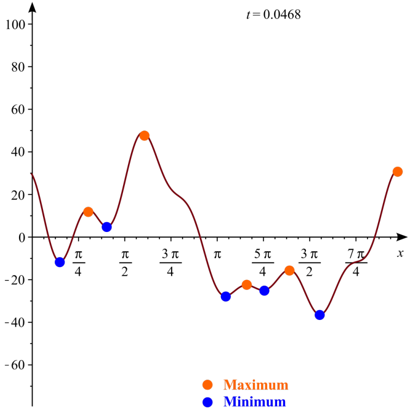

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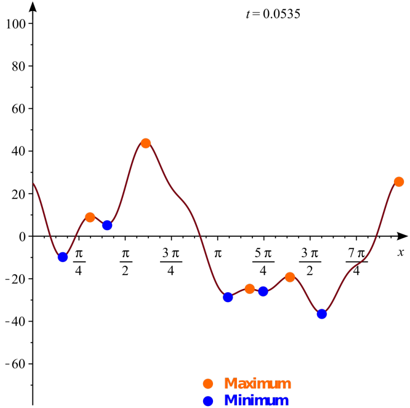

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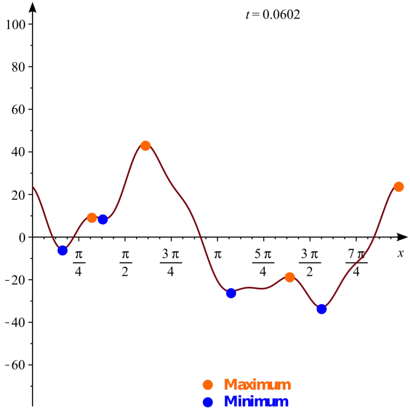

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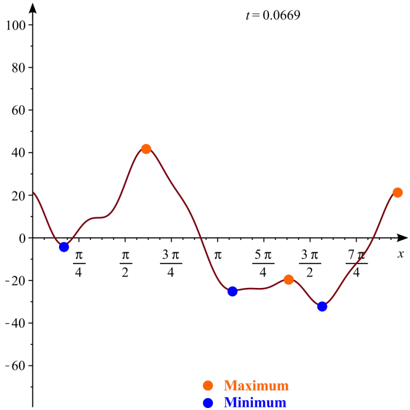

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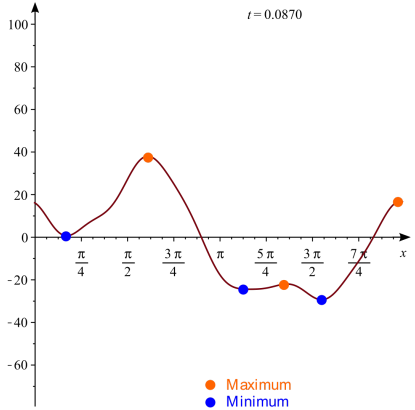

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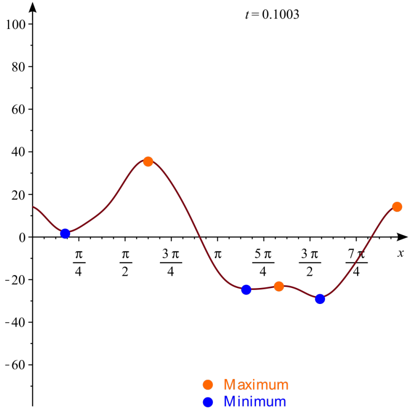

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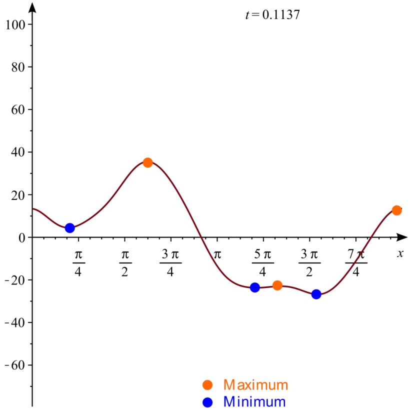

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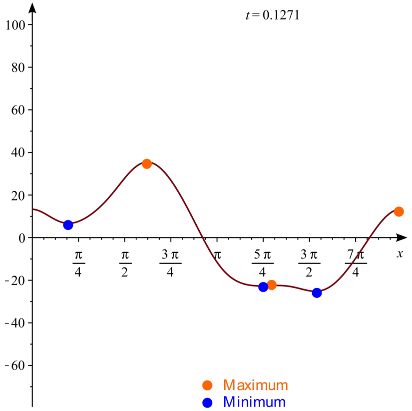

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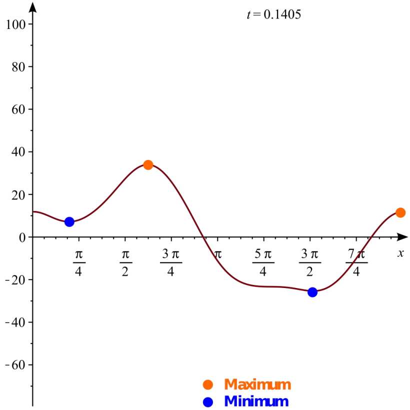

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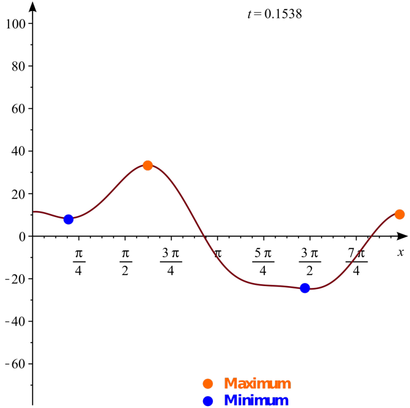

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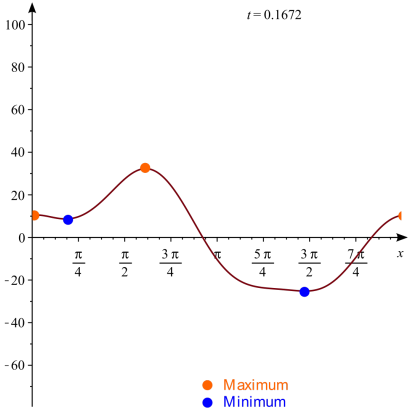

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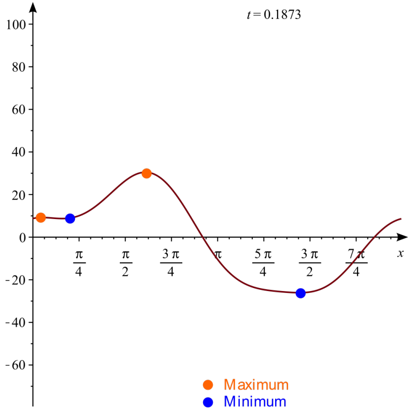

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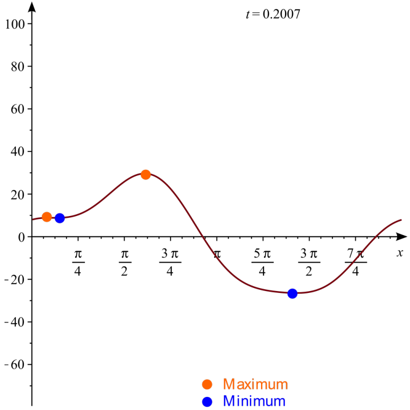

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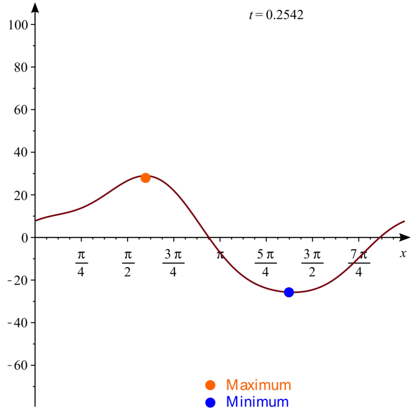
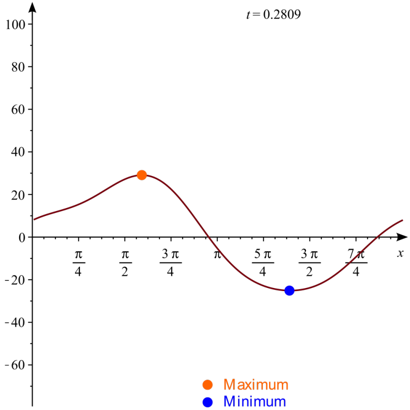


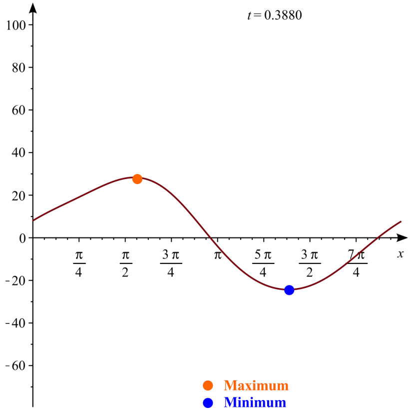
Circle S^1

$$f(x,t) = c + [a_1 \sin(x) + b_1 \cos(x)]\, e^{-t} + [a_2 \sin(2x) + b_2 \cos(2x)]\, e^{-4t} + [a_3 \sin(3x) + b_3 \cos(3x)]\, e^{-9t} + [a_4 \sin(4x) + b_4 \cos(4x)]\, e^{-16t} + \cdots$$



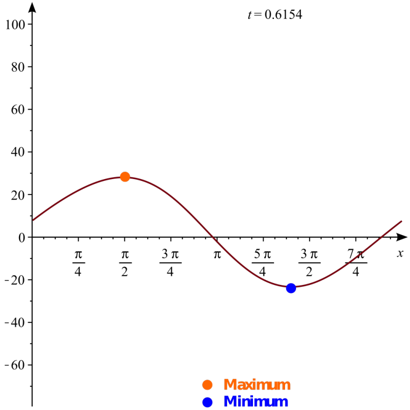
Circle S^1

$$f(x,t) = c + [a_1 \sin(x) + b_1 \cos(x)]\, e^{-t} + [a_2 \sin(2x) + b_2 \cos(2x)]\, e^{-4t} + [a_3 \sin(3x) + b_3 \cos(3x)]\, e^{-9t} + [a_4 \sin(4x) + b_4 \cos(4x)]\, e^{-16t} + \cdots$$



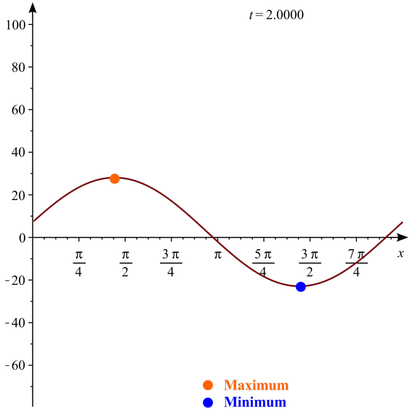
Circle S^1

$$
\begin{aligned}
f(x,t) = & c + [a_1 \sin(x) + b_1 \cos(x)]\, e^{-t} + [a_2 \sin(2x) + b_2 \cos(2x)]\, e^{-4t} \\
& + [a_3 \sin(3x) + b_3 \cos(3x)]\, e^{-9t} + [a_4 \sin(4x) + b_4 \cos(4x)]\, e^{-16t} \\
& + \cdots
\end{aligned}
$$



Circle S^1

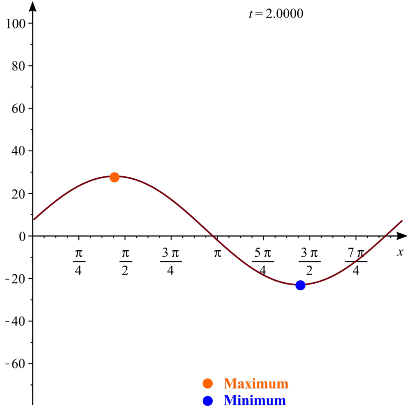
$$f(x,t) = c + [a_1 \sin(x) + b_1 \cos(x)]\, e^{-t} + [a_2 \sin(2x) + b_2 \cos(2x)]\, e^{-4t} + [a_3 \sin(3x) + b_3 \cos(3x)]\, e^{-9t} + [a_4 \sin(4x) + b_4 \cos(4x)]\, e^{-16t} + \cdots$$



Circle S^1

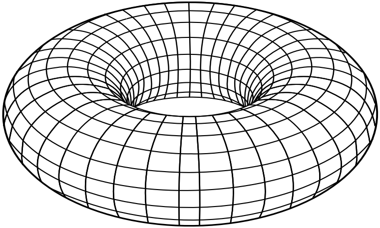
$$f(x,t) = c + [a_1 \sin(x) + b_1 \cos(x)]\, e^{-t} + [a_2 \sin(2x) + b_2 \cos(2x)]\, e^{-4t} + [a_3 \sin(3x) + b_3 \cos(3x)]\, e^{-9t} + [a_4 \sin(4x) + b_4 \cos(4x)]\, e^{-16t} + \cdots$$

◂ Initiation



Tori T^2

(Cadavid and Velez, 2003)

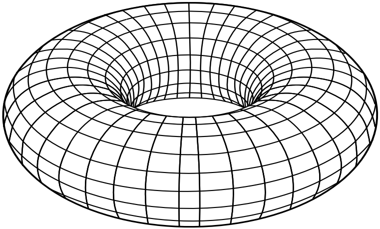


Tori T^2

(Cadavid and Velez, 2003)

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 4, \quad \lambda_4 = 5, \cdots$$

$$\begin{aligned} E_{\lambda_1} &= span\left\{\sin(x), \cos(x), \sin(y), \cos(y)\right\} \\ E_{\lambda_2} &= span\left\{\sin(x+y), \cos(x+y)\right\} \\ E_{\lambda_3} &= span\left\{\sin(2x), \cos(2x), \sin(2y), \cos(2y)\right\} \\ E_{\lambda_4} &= span\left\{\sin(2x+y), \cos(x+2y)\right\} \\ &\vdots \end{aligned}$$

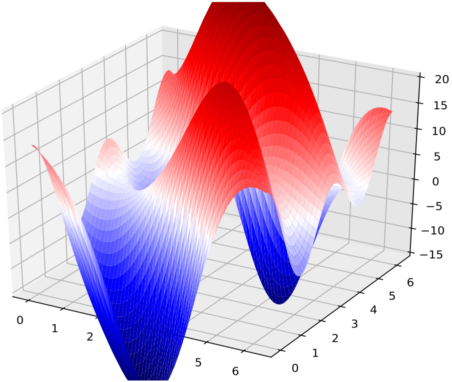


Tori T^2

$$
\begin{aligned}
f(x,y,t) = & c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\
& + \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\
& + \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots
\end{aligned}
$$

Tori T^2

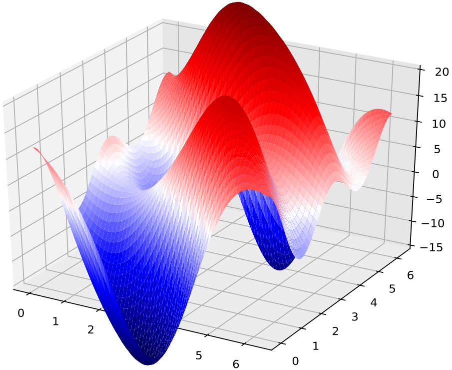
$$
\begin{aligned}
f(x,y,t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
& + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
& + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.05$

Tori T^2

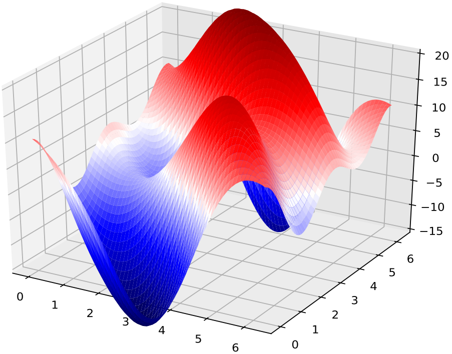
$$\begin{aligned} f(x,y,t) =& c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\ &+ \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\ &+ \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots \end{aligned}$$



$t = 0.10$

Tori T^2

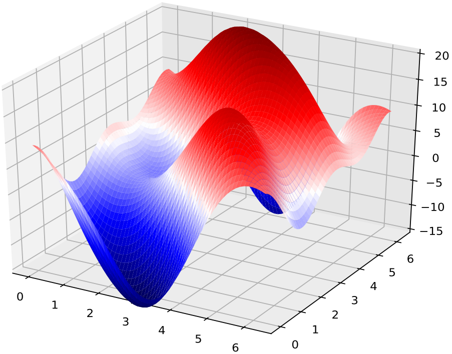
$$
\begin{aligned}
f(x,y,t) =& c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\
&+ \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\
&+ \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.15$

Tori T^2

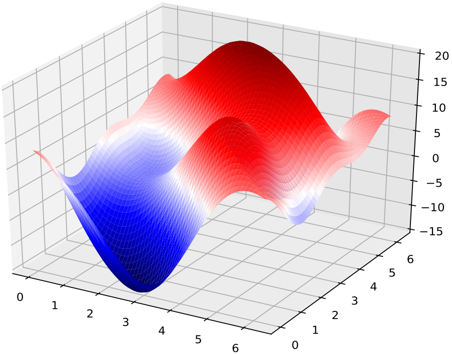
$$
\begin{aligned}
f(x,y,t) =& c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
&+ [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
&+ [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.20$

Tori T^2

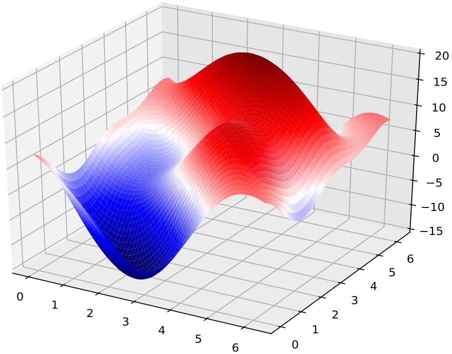
$$
\begin{aligned}
f(x,y,t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
& + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
& + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.25$

Tori T^2

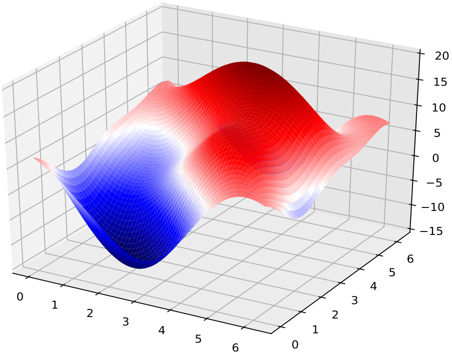
$$
\begin{aligned}
f(x,y,t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
& + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
& + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.30$

Tori T^2

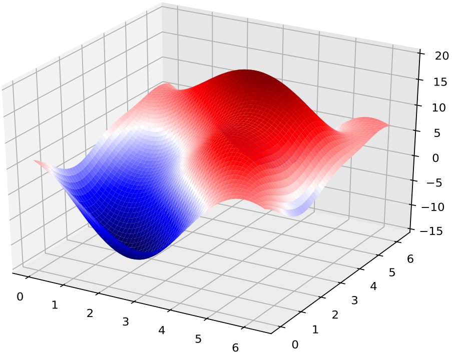
$$
\begin{aligned}
f(x, y, t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
& + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
& + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.35$

Tori T^2

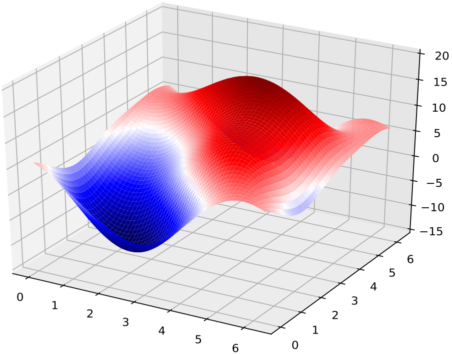
$$
\begin{aligned}
f(x,y,t) =& c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\
&+ \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\
&+ \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.40$

Tori T^2

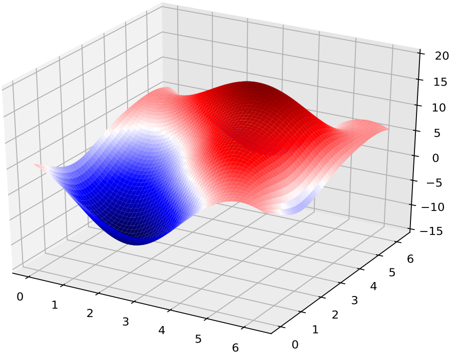
$$f(x,y,t) = c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t}$$
$$+ [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t}$$
$$+ [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots$$



$t = 0.45$

Tori T^2

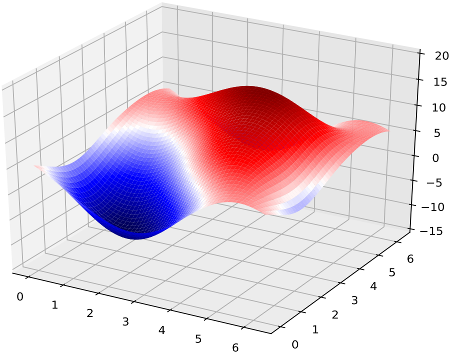
$$\begin{aligned} f(x,y,t) =& c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\ &+ [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\ &+ [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots \end{aligned}$$



$t = 0.50$

Tori T^2

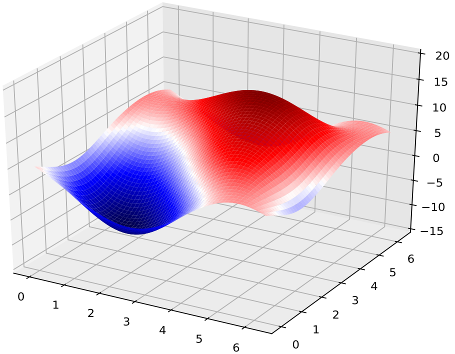
$$\begin{aligned} f(x,y,t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\ & + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\ & + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots \end{aligned}$$



$t = 0.55$

Tori T^2

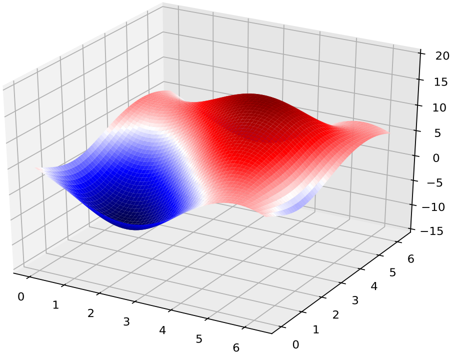
$$
\begin{aligned}
f(x,y,t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
& + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
& + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.60$

Tori T^2

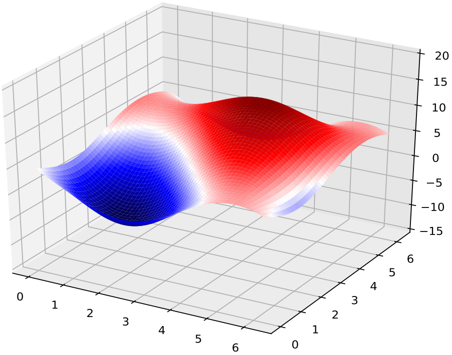
$$\begin{aligned} f(x,y,t) =& c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\ &+ [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\ &+ [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots \end{aligned}$$



$t = 0.65$

Tori T^2

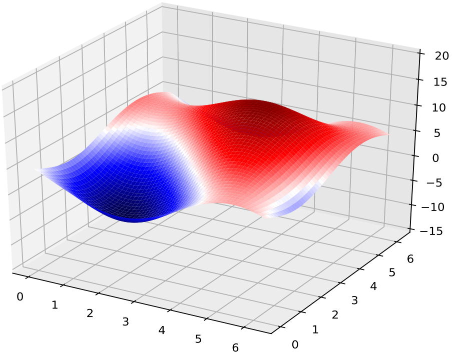
$$\begin{aligned} f(x,y,t) =& c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\ &+ \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\ &+ \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots \end{aligned}$$



$t = 0.70$

Tori T^2

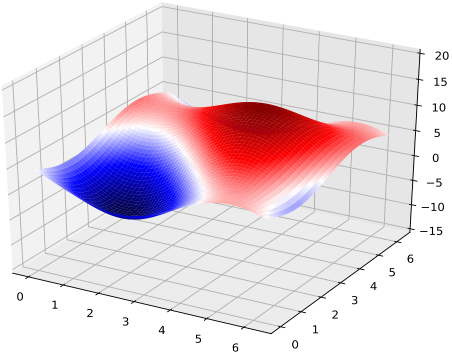
$$\begin{aligned} f(x,y,t) =& c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\ &+ \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\ &+ \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots \end{aligned}$$



$t = 0.75$

Tori T^2

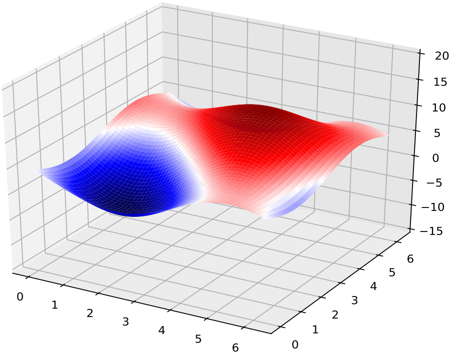
$$
\begin{aligned}
f(x,y,t) =& c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\
&+ \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\
&+ \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.80$

Tori T^2

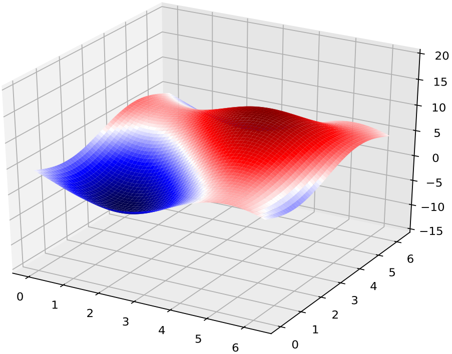
$$
\begin{aligned}
f(x,y,t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
& + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
& + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.85$

Tori T^2

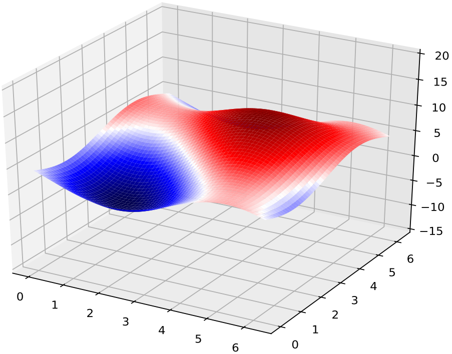
$$
\begin{aligned}
f(x,y,t) =& c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
&+ [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
&+ [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.90$

Tori T^2

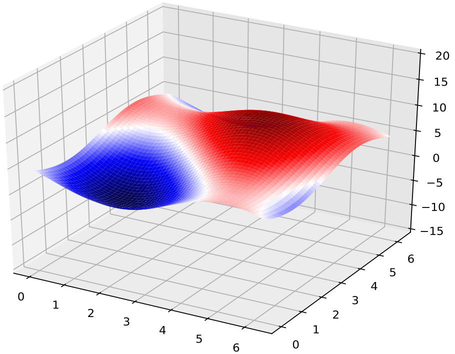
$$
\begin{aligned}
f(x,y,t) =& c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\
& + \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\
& + \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots
\end{aligned}
$$



$t = 0.95$

Tori T^2

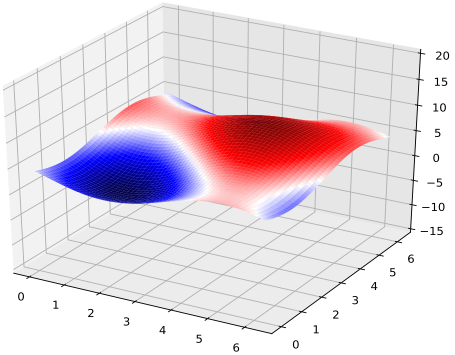
$$
\begin{aligned}
f(x,y,t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
& + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
& + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 1.00$

Tori T^2

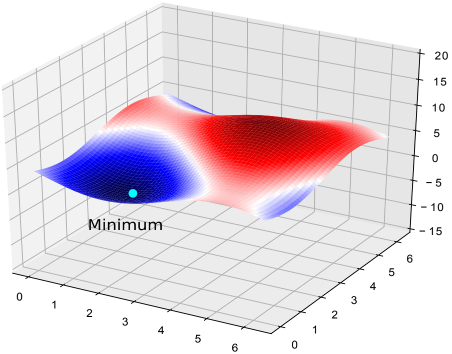
$$\begin{aligned} f(x,y,t) = & c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\ & + [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\ & + [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots \end{aligned}$$



$t = 1.2$

Tori T^2

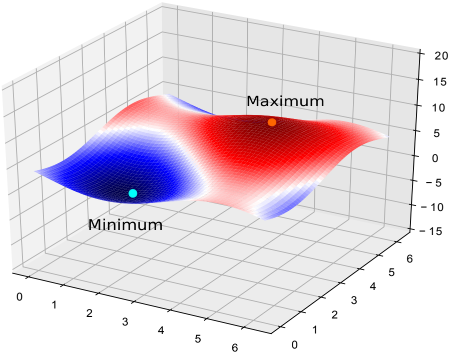
$$
\begin{aligned}
f(x,y,t) =& c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\
&+ [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\
&+ [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots
\end{aligned}
$$



$t = 1.2$

Tori T^2

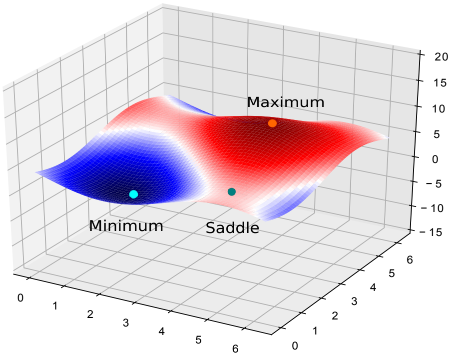
$$\begin{aligned} f(x,y,t) =& c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\ &+ [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\ &+ [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots \end{aligned}$$



$t = 1.2$

Tori T^2

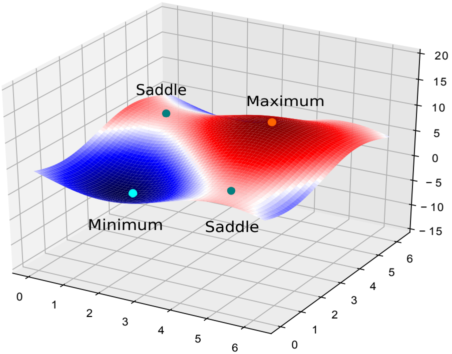
$$\begin{aligned} f(x,y,t) =& c + [a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)]\, e^{-t} \\ &+ [a_2 \sin(x+y) + b_2 \cos(x+y)]\, e^{-2t} \\ &+ [a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)]\, e^{-4t} + \cdots \end{aligned}$$



$t = 1.2$

Tori T^2

$$
\begin{aligned}
f(x,y,t) =& c + \left[a_1 \sin(x) + b_1 \cos(x) + c_1 \sin(y) + d_1 \cos(y)\right] e^{-t} \\
&+ \left[a_2 \sin(x+y) + b_2 \cos(x+y)\right] e^{-2t} \\
&+ \left[a_3 \sin(2x) + b_3 \cos(2x) + c_3 \sin(2y) + d_3 \cos(2y)\right] e^{-4t} + \cdots
\end{aligned}
$$



$t = 1.2$

The Hantzsche-Wendt space

Take $E^3 = \left(\mathbb{R}^3, g_e\right)$ where g_e is the euclidean geometry.
Let G be the group of isometries of E^3 generated by

$$\begin{aligned} f :&(x, y, z) \to (x+2, -y+2, -z) \\ g :&(x, y, z) \to (-x, y+2, -z+2) \\ h :&(x, y, z) \to (-x+2, -y, z+2) \end{aligned}$$

The Hantzsche-Wendt space is E^3/G

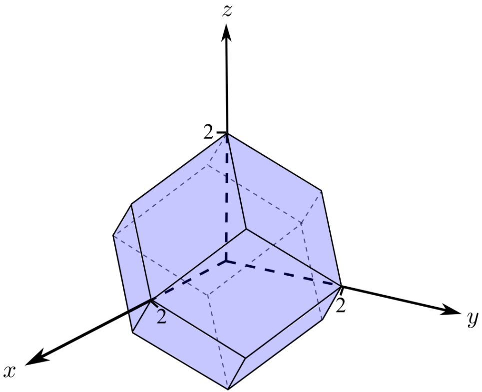
Hantzsche-Wendt E^3/G

Hantzsche-Wendt E^3/G

A fundamental domain for The Hantzsche-Wendt space is:

Hantzsche-Wendt E^3/G

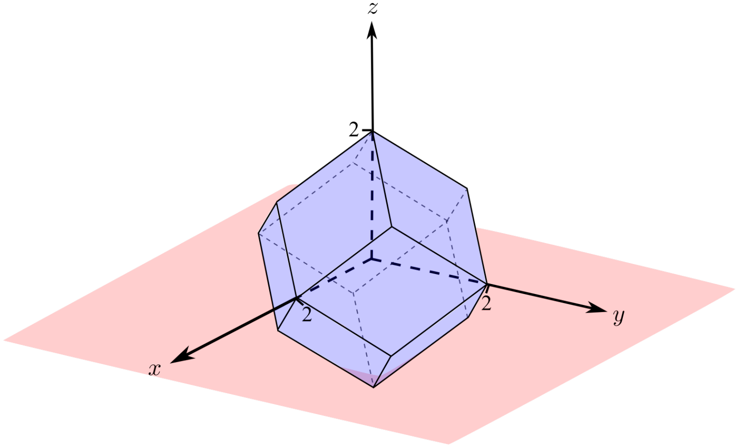
A fundamental domain for The Hantzsche-Wendt space is:



Rhombic dodecahedron

Hantzsche-Wendt E^3/G

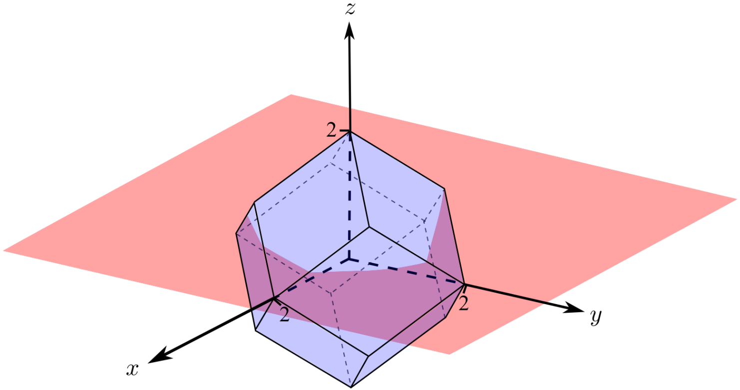
A fundamental domain for The Hantzsche-Wendt space is:



Rhombic dodecahedron

Hantzsche-Wendt E^3/G

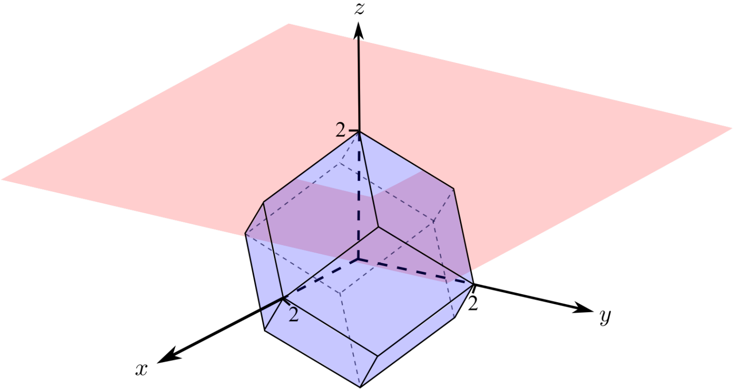
A fundamental domain for The Hantzsche-Wendt space is:



Rhombic dodecahedron

Hantzsche-Wendt E^3/G

A fundamental domain for The Hantzsche-Wendt space is:



Rhombic dodecahedron

(Riazuelo et al., 2004)
E_{λ_1} is spanned by

$$\sin\left(\frac{\pi}{2}(x+y)\right)+\sin\left(\frac{\pi}{2}(x-y)\right),$$
$$\sin\left(\frac{\pi}{2}(y+z)\right)+\sin\left(\frac{\pi}{2}(y-z)\right),$$
$$\sin\left(\frac{\pi}{2}(x+z)\right)-\sin\left(\frac{\pi}{2}(x-z)\right),$$

E_{λ_2} is spanned by

$$\cos\left(\frac{\pi}{2}(x+y+z)\right)+\cos\left(\frac{\pi}{2}(x-y-z)\right)+\cos\left(\frac{\pi}{2}(x-y+z)\right)+$$
$$\cos\left(\frac{\pi}{2}(x+y-z)\right),$$
$$\sin\left(\frac{\pi}{2}(x+y+z)\right)+\sin\left(\frac{\pi}{2}(x-y-z)\right)-\sin\left(\frac{\pi}{2}(x-y+z)\right)-$$
$$\sin\left(\frac{\pi}{2}(x+y-z)\right),$$

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
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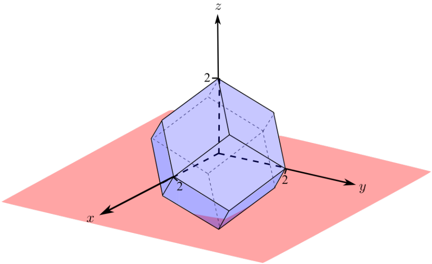
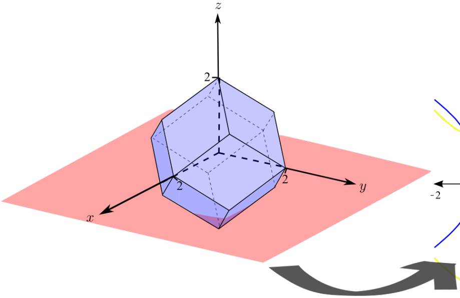
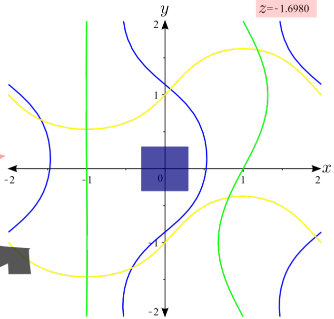


Hantzsche-Wendt E^3/G

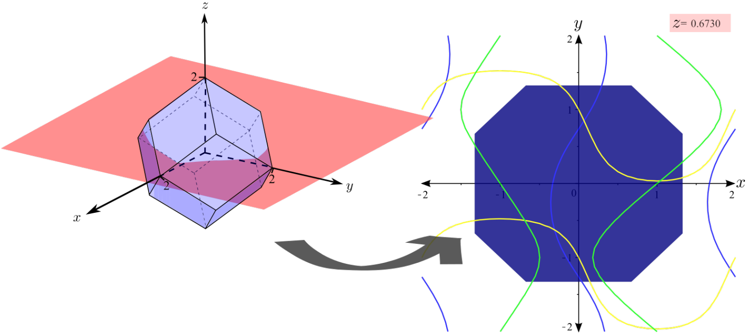
$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$





Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

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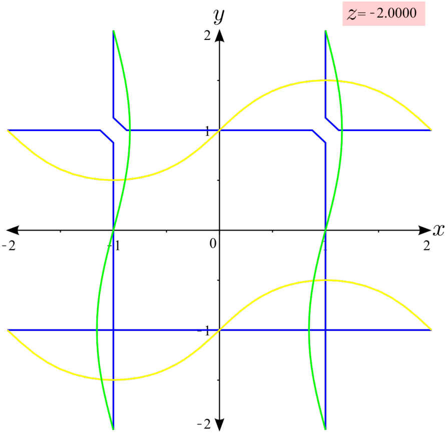
Hantzsche-Wendt E^3/G

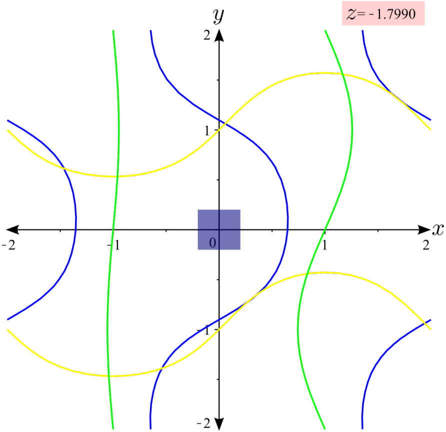
$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



Critical points

Hantzsche-Wendt E^3/G

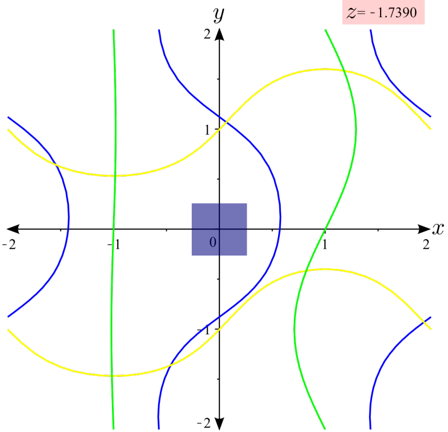
$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



Critical points

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



Critical points

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
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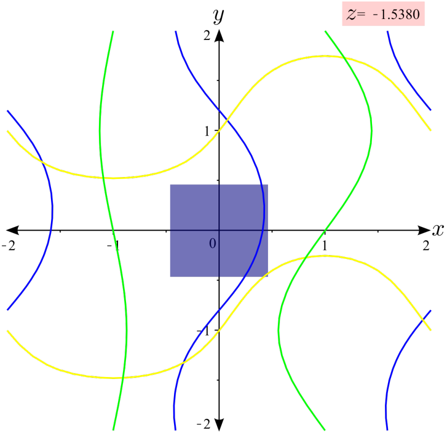
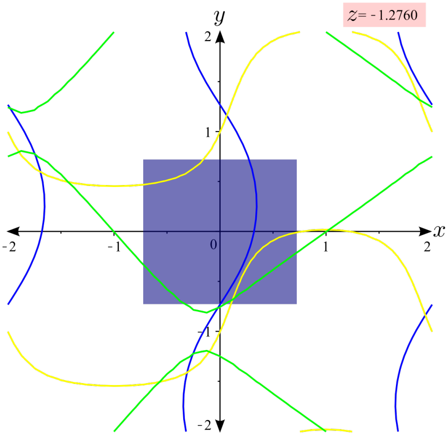


Critical points

Hantzsche-Wendt E^3/G

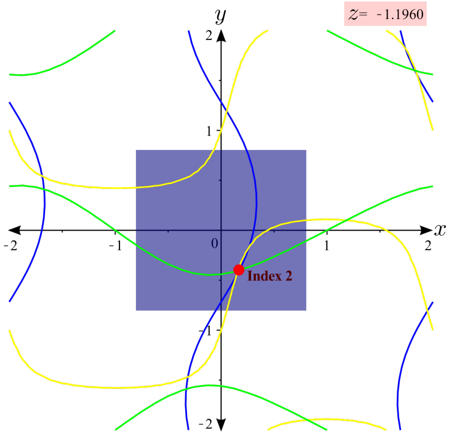
$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



Critical points

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

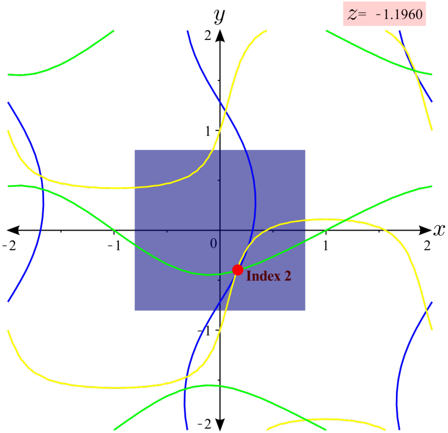


Critical points

Index 2

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

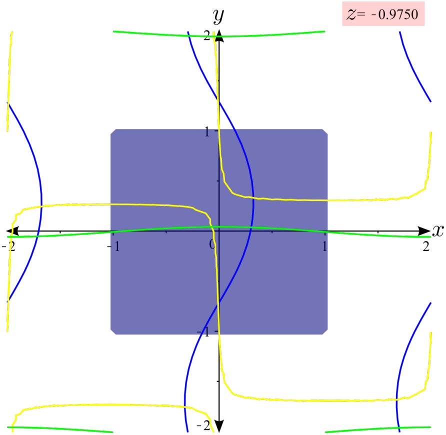


Critical points

Index 2

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

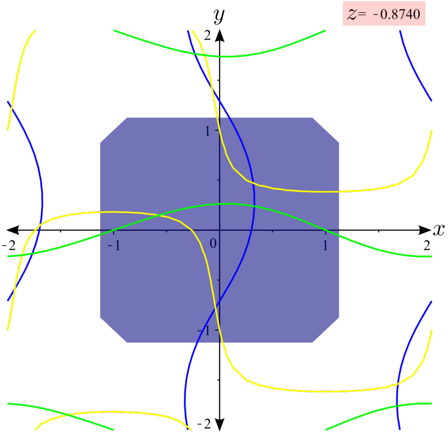


Critical points

Index 2

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

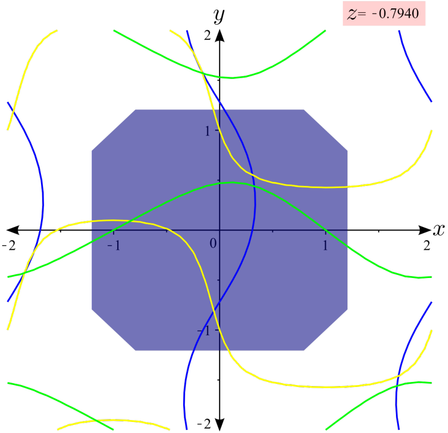


Critical points

Index 2

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

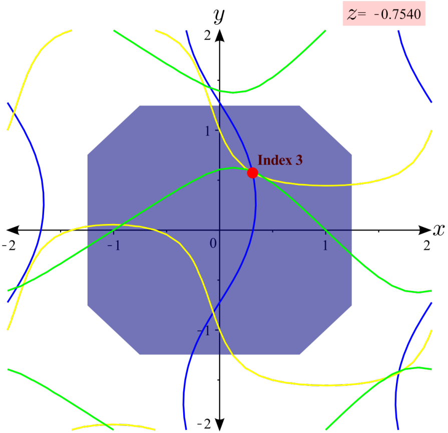


Critical points

Index 2

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



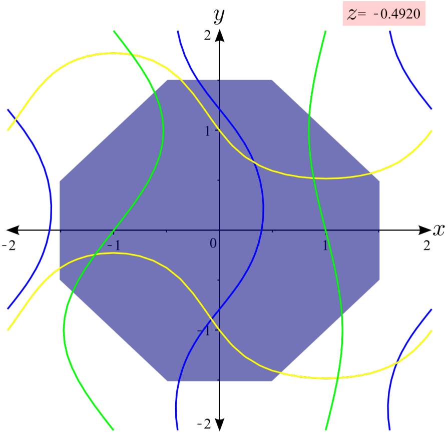
Critical points

Index 2

Index 3

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



Critical points

Index 2

Index 3

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
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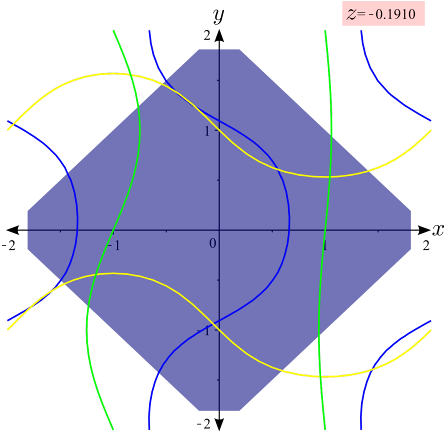


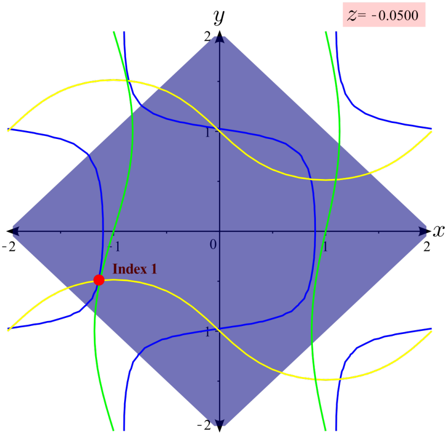
Critical points

Index 2

Index 3

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

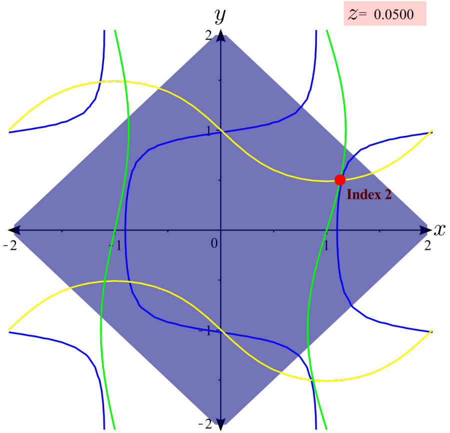


Critical points

Index 2
Index 3
Index 1

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

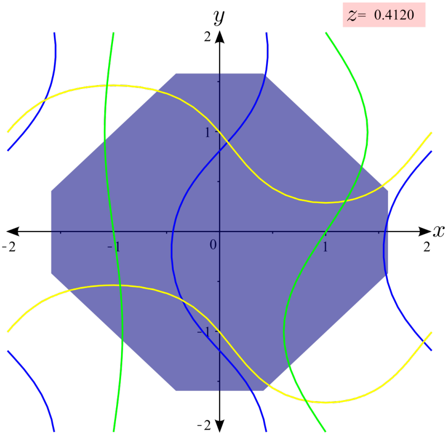


Critical points

Index 2
Index 3
Index 1
Index 2

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$-2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

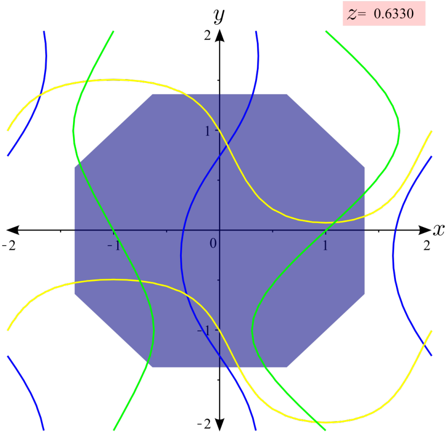


Critical points

Index 2
Index 3
Index 1
Index 2

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

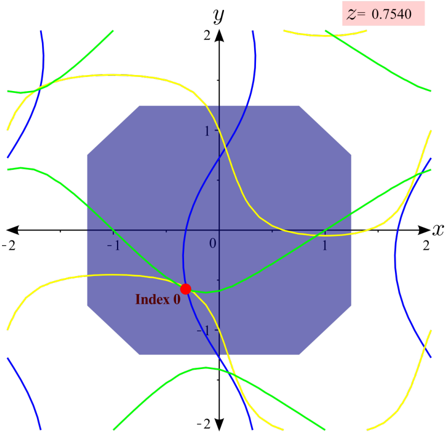


Critical points

Index 2
Index 3
Index 1
Index 2

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

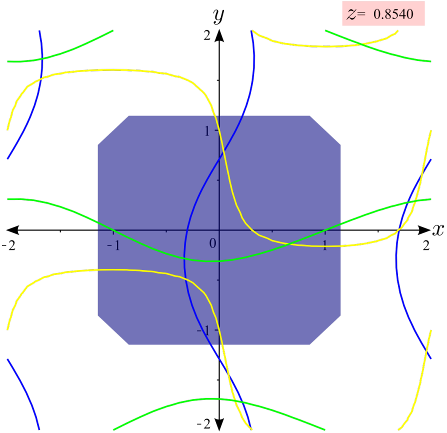


Critical points

Index 2
Index 3
Index 1
Index 2
Index 0

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



Critical points

Index 2
Index 3
Index 1
Index 2
Index 0

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
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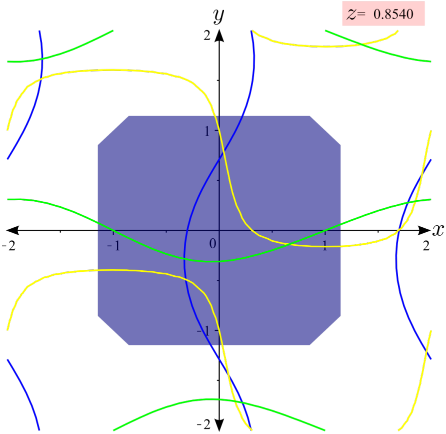
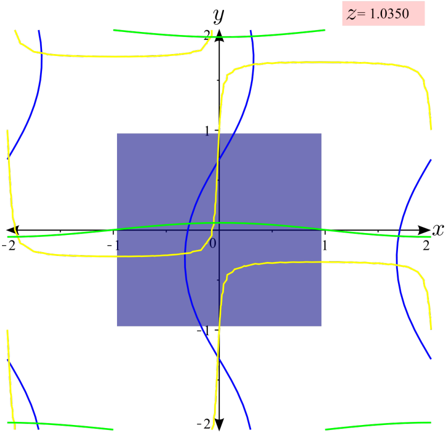


Critical points

Index 2
Index 3
Index 1
Index 2
Index 0

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
$$- 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

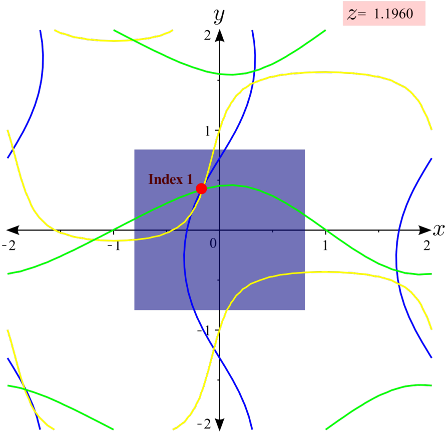


Critical points

Index 2
Index 3
Index 1
Index 2
Index 0

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$

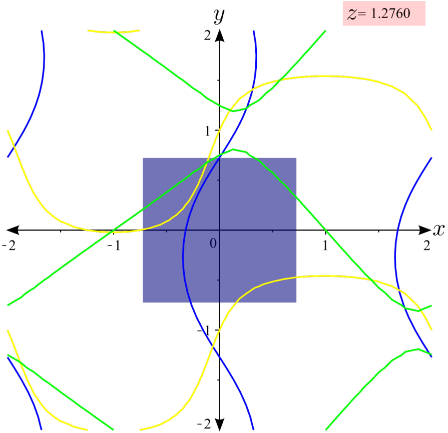


Critical points

Index 2
Index 3
Index 1
Index 2
Index 0
Index 1

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right] - 2\left[\sin\left(\frac{\pi}{2}(x+z)\right) - \sin\left(\frac{\pi}{2}(x-z)\right)\right]$$



Critical points

Index 2
Index 3
Index 1
Index 2
Index 0
Index 1

Hantzsche-Wendt E^3/G

$$f = \left[\sin\left(\frac{\pi}{2}(x+y)\right) + \sin\left(\frac{\pi}{2}(x-y)\right)\right] + \left[\sin\left(\frac{\pi}{2}(y+z)\right) + \sin\left(\frac{\pi}{2}(y-z)\right)\right]$$
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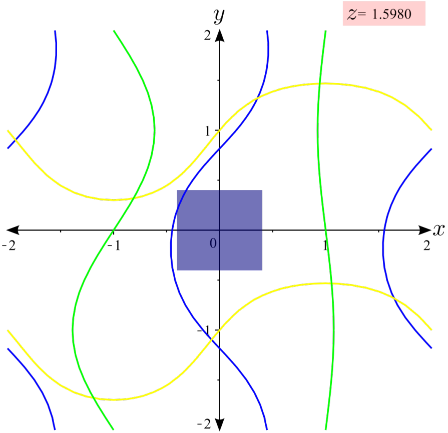

Critical points

Index 2
Index 3
Index 1
Index 2
Index 0
Index 1

Hantzsche-Wendt E^3/G

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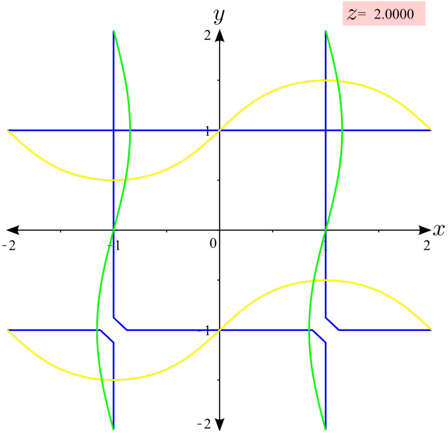

Critical points

Index 2
Index 3
Index 1
Index 2
Index 0
Index 1
Total 6

This has been proved for:

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- Round spheres of all dimensions

$$S^n = \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}, n \geq 1$$

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- Flat Klein bottle
- Spherical projective plane
- Complex projective spaces endowed with the Fubini-Study metric

There is strong experimental evidence that the phenomenon takes place for:

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- Spherical dodecahedral Poincaré space

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THANKS!