Post-Quantum Cryptography

Based on Lattices and on Multivariate Polynomial Equations

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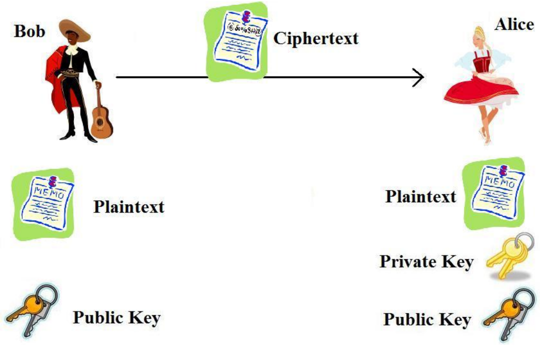
Overview

Post-Quantum Cryptography

- Public key crypto – 60's – DH (Discrete log), RSA (Factoring)



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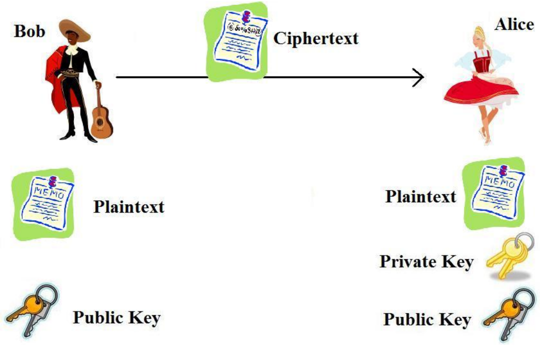

- Shor's discrete log and factoring quantum algorithms 1996
- First post-quantum workshop 2006
- NSA anouncement 2015, NIST competition open 2016

Post-Quantum Cryptography – Flavors

Based on:

- Lattice theory
- Multivariate Polynomials
- Coding Theory
- Hash functions

Content

Lattice-Based Crypto

Pros:

- Robust security guarantee: Worst-to-average-case reduction
- Flexibility: Homomorphic encryption, obfuscation

Cons:

- Inefficient
- Hard to determine secure parameters

Cryptographic Hash Function

A cryptographic hash function $h(x)$ provides:

- **Compression:** the range of h is smaller than its domain
- **Efficiency:** It is efficient to compute h
- **Collision resistance:** It is unfeasible to find $x \neq y$ such that $h(x) = h(y)$

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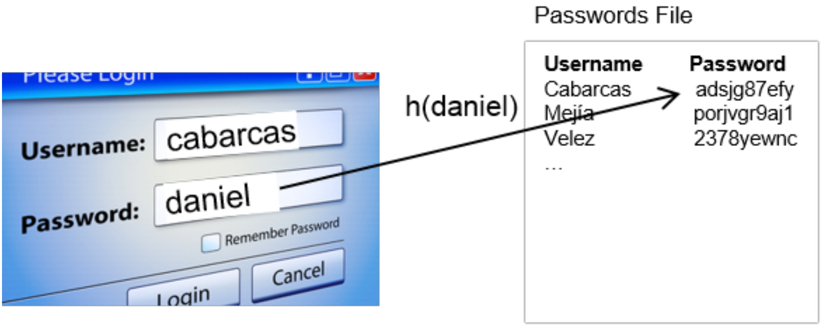
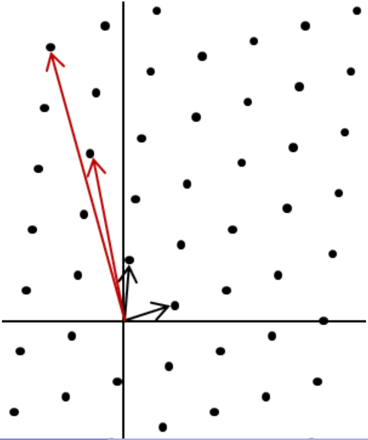
- **Application:** Store passwords



Lattice Theory

- A **lattice** is a discrete subgroup of $\mathbb{R}^n$
- A **basis** B for a lattice L is a set of LI vectors $b_1, \ldots, b_m$ such that $L = \mathcal{L}(B) = \{\sum x_i b_i : x_i \in \mathbb{Z}\}$

- Given an arbitrary basis, in general, it is hard to find **the shortest vector**
- The best algorithms are:
  - LLL, finds an exp. approx. in poly time
  - Ennumeration, finds the shortest vector in exp. time
  - BKZ, can be tuned



Lattice-based Hash [Ajt96]

- For a security parameter n
- Let $m, q \in \mathbb{N}$ be such that $n \log q < m < \frac{q}{2n^4}$ y $q = \mathcal{O}(n^c)$ for some constant c (e.g. $m = n^2, q = n^7$)
- Choose $M \leftarrow \mathbb{Z}_q^{n \times m}$

$$\begin{aligned} h_M \colon \{0,1\}^m &\to \mathbb{Z}_q^n \\ s &\mapsto Ms \mod q \end{aligned}$$

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- Compression and efficiency are easy to verify
- Security: consider the lattice

$$\Lambda_q^*(M) = \{v \in \mathbb{R}^m : Mv = 0 \mod q\}$$

Worst-to-Average-Case Reduction

Theorem ([Ajt96])

If it is possible for an adversary, that executes in poli(n) *time, to find a collision of* **one** *randomly chosen instance of* h_M *with non-negligible probability, then it is possible to solve* **any** *instance of* n^c*-SIVP in dimension* n *in* poli(n) *time.*

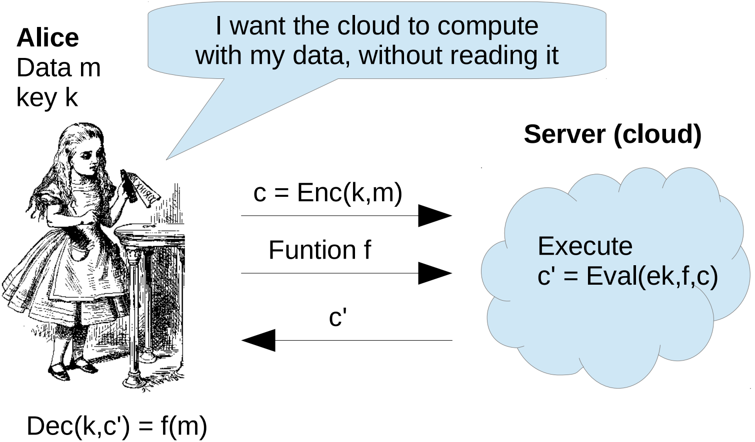
Developments Since Ajtai's work

- Efficiency improvements to Ajtai's hash [Mic01, MR07]
- LWE problem and public key encryption based on LWE [Reg05]
- Ring-LWE [LPR10]
- Fully homomorphic encryption (FHE) [Gen09]

My Work

- Discrete Ziggurat Gaussian Sampling [BCG+14]
- Uniform noise lattice-based encryption [CGW14]
- Ring Isomorphism encoding for FHE [GC14]

Homomorphic Encryption



Integer Encoding in HE

- The potential of HE is huge
- To compute over encrypted data is **ineficient**
- We should take the most out of each homomorphic operation
- The space of messages in HE has an **algebraic structure** e.g $R_p = \mathbb{Z}[x]/\langle x^n + 1, p\rangle$, with p an integer

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¿How to encode as much information as possible in this message space, in a way that the operations are meaningful?

RIE Integer Encoding [GC14]

- Message space: $R_p = \mathbb{Z}[x]/\left\langle x^n + 1, p \right\rangle$

Key observation: R_p is isomorphic as a ring to $\mathbb{Z}_t$ for certain polynomials p

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- Then we can encode integers directly
- This does not affect the security of the scheme
- And improves the efficency compared to previously used encodings

Content

Crypto Based on Multivariate Polynomials

Pros:

- Efficient constructions
- Practical security well understood

Cons:

- Attacks on Ad hoc constructions have deslegitimize the area

MPKC – General Idea

- k finite field of q elements (e.g., $k = \{0, 1\}$).

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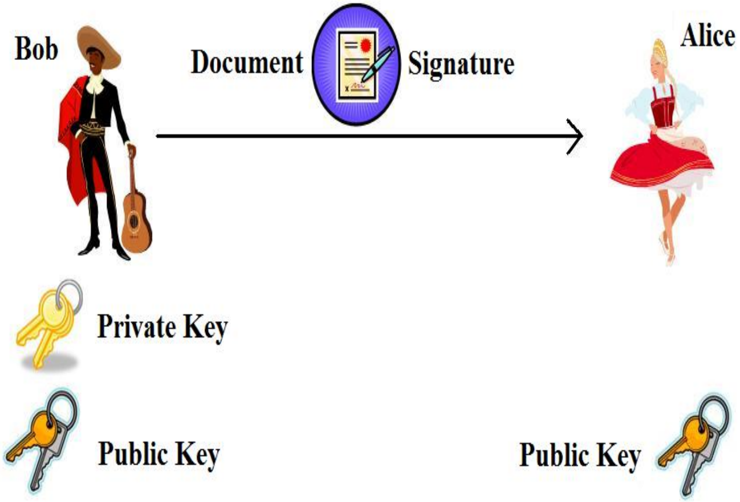
- **Plaintext:** $(x_1, \ldots, x_n)$.
- **Ciphertext:** $(y_1, \ldots, y_m) = P(x_1, \ldots, x_n)$.
- **Underlying problem:**
  - MQ: Solving a system of multivariate quadratic equations
  - NP-complete

Signature Scheme



MPKC Signature Scheme – Oil-Vinegar [Pat97]

- Oil-Vinegar polynomial:

$$f = \sum_{i=1}^{o}\sum_{j=1}^{v} a_{ij}x_i t_j + \sum_{i=1}^{v}\sum_{j=1}^{v} b_{ij}t_i t_j + \sum_{i=1}^{o} c_i x_i + \sum_{j=1}^{v} d_j t_j + e,$$

with $a_{ij}, b_{ij}, c_i, d_j, e \in k$.

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- **Private key:**
  - $F = (f_1, \ldots, f_o) : k^{o+v} \to k^o$, with f_i randomly chosen O-V polynomials.
  - $L : k^{o+v} \to k^{o+v}$ invertible affine transformation chosen uniformly at random.

- **Public key:** $P = (p_1, \cdots, p_o) : k^{o+v} \to k^o$ defined by

$$P(x_1, \ldots, x_o, t_1, \ldots, t_v) = F \circ L(x_1, \ldots, x_o, t_1, \ldots, t_v).$$

Oil-Vinegar Signature Generation

- Randomly choose $t_1 = w_1, \ldots, t_v = w_v$, with $w_1, \ldots, w_v \in k$.

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- Plug those into $F \Rightarrow$ linear system in variables $x_1, \dots, x_o$:

$$\begin{aligned} f_1(x_1, \dots, x_o, w_1, \dots, w_v) &= \tilde{y}_1 \\ f_2(x_1, \dots, x_o, w_1, \dots, w_v) &= \tilde{y}_2 \\ &\vdots \\ f_o(x_1, \dots, x_o, w_1, \dots, w_v) &= \tilde{y}_o, \end{aligned} \tag{1}$$

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- We solve this system using Gaussian elimination.
- If (1) has no solutions, repeat the process.

Developments in MPKC

- Signature schemes: U-OV [KPG99], HFEv- [PGC98]
- Identification scheme: Sakumoto [SSH11]
- Stream cipher: QUAD [BGP06]
- Attacks: Groebner bases [FJ03], min-rank [KS99], differential [DFSS07]

My Work

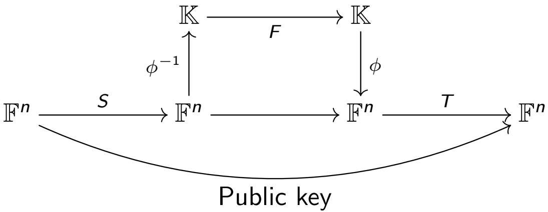
- Improvements in Groebner basis computation [MCD+10, BCDM10, CD11]
- Improvements to ZHFE [BCE+16]

Hidden Field Equations (HFE)

- $\mathbb{F} = \mathbb{F}_q$, $\mathbb{K}$ a degree n extension field of $\mathbb{F}$.
- $\phi\colon \mathbb{K} \to \mathbb{F}^n$ the standard $\mathbb{F}$-linear isomorphism.
- Choose a polynomial $F : \mathbb{K} \to \mathbb{K}$ of Hamming weight two, i.e,

$$F(X) = \sum_{q^i+q^j \leq D} a_{ij} X^{q^i+q^j} + \sum_{q^i \leq D} b_i X^{q^i} + c,$$

- Choose uniformly at random two invertible affine transformations S and T over $\mathbb{F}^n$.



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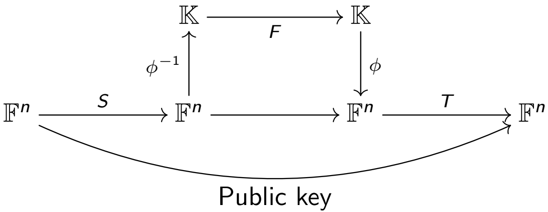
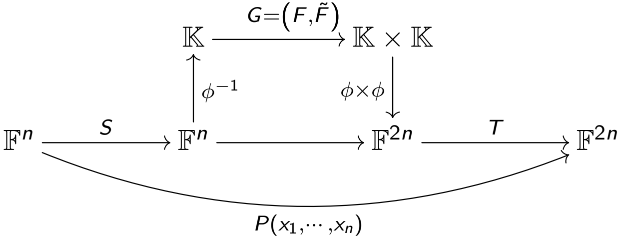


- Broken by Groebner bases [FJ03], and min-rank [KS99]

ZHFE trapdoor function [PBD15]

- $G = \left(F, \tilde{F}\right) : \mathbb{K} \to \mathbb{K} \times \mathbb{K}$ with $F, \tilde{F}$ of high degree and hegh rank.
- The new trapdoor function is $P = T \circ (\phi \times \phi) \circ G \circ \phi^{-1} \circ S$.



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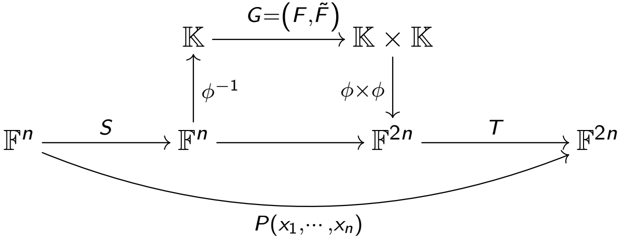

- G is chosen so that there exist Ψ of the form

$$\begin{aligned}\Psi =& X\left(\alpha_1 F_0 + \cdots + \alpha_n F_{n-1} + \beta_1 \tilde{F}_0 + \cdots + \beta_n \tilde{F}_{n-1}\right) + \\ & X^q\left(\alpha_{n+1} F_0 + \cdots + \alpha_{2n} F_{n-1} + \beta_{n+1} \tilde{F}_0 + \cdots + \beta_{2n} \tilde{F}_{n-1}\right),\end{aligned}$$

  such that $\deg(\Psi) \leq D_0$.

Thanks

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Appendix 1: SIVP

n^c-SIVP. Approximate Shortest Independent Vectors Problem

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Given a basis B for an n dimension lattice L in $\mathbb{R}^n$, find a set of n LI vectors $v_1, \ldots, v_n$ such that

$$\max\{\|v_i\|\} \leq n^c \cdot \min\{\|S\| : S \text{ Set of } n \text{ LI vectors in } L\}$$